Vanishing Ideals for Codes on Toric Varieties

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Motivated by applications to the theory of error-correcting codes, we give an algorithmic method for computing a generating set for the ideal generated by β-graded polynomials vanishing on a subset of a simplicial complete toric variety X over a finite field $\mathbb{F}_q$, parameterized by rational functions, where β is a $d \times r$ matrix whose columns generate a subsemigroup $\mathbb{N}\beta$ of $\mathbb{N}^d$. We also give a method for computing the vanishing ideal of the set of $\mathbb{F}_q$-rational points of X. We talk about some of its algebraic invariants related to basic parameters of the corresponding evaluation code. When $\beta = [w_1 \cdots w_r]$ is a row matrix corresponding to a numerical semigroup $\mathbb{N}\beta = \langle w_1, \ldots, w_r \rangle$, X is a weighted projective space and generators of its vanishing ideal is related to the generators of the defining (toric) ideal of the numerical semigroup rings corresponding to semigroups generated by subsets of $\{w_1, \ldots, w_r\}$.

Toric codes, introduced for the first time by Hansen [3], are obtained by evaluating all homogeneous polynomials of degree $\alpha \in \mathbb{N}\beta$ at only the $\mathbb{F}_q$-rational points of the dense torus $T_X \subset X$. They are studied extensively, see [4, 6] and references therein. Some record breaking examples are found replacing the vector space S_α, consisting of all homogeneous polynomials of degree α, by its subspaces, see [2]. In another direction, Nardi offered to extend the length of a toric code by evaluating at the full set of $\mathbb{F}_q$-rational points $X(\mathbb{F}_q)$ in [5] which revealed importance of computing a generating set for the vanishing ideal of $X(\mathbb{F}_q)$, see also [1].

References

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